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# THE PROBLEM OF CONFLICTING CONTROL WITH MIXED CONSTRAINTS†

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The control problem for a linear dynamical system is considered at a minimax of the terminal quality index. Feasible controls are simultaneously restricted by geometrical constraints and by integrated momentum constraints, the latter being thought of as a store of control resources. The problem is formalized as a differential game [1–4] using concepts [5–8] developed at Ekaterinburg. Here, because of the geometrical constraints, the momentum formulation and its associated difficulties [2–4] do not appear. On the other hand the presence of the integral restrictions leads to the appearance of additional variables whose evolution describes the dynamics of the expenditure of the control resources. These variables are subject to phase restrictions, which is a peculiarity of the problem. A reasonably informative picture and a class of strategies for which the given game has a value and a saddle point are given. A constructive method for computing the value function of the game and constructing optimal strategies is presented. This method is conceptually related to the construction of a stochastic programming synthesis [5] and is based on the recursive construction of upper-convex envelopes for certain auxiliary functions. The possibility of exchanging the minimum and maximum operations over the resource parameters when calculating the value of the game using these procedure is established.

The present paper extends the constructions described in [5–7], in accordance with [8], to problems with mixed restrictions on the controls.

## 1. STATEMENT OF THE PROBLEM

Suppose that the evolution of a dynamical system is described by the equation

$$dx/dt = A(t)x + B(t)u + C(t)\upsilon \tag{1.1}$$
$$x \in R^n, \quad u \in R^r, \quad \upsilon \in R^s, \quad t_0 \leqslant t \leqslant \vartheta$$

where $x$ is the phase vector, $u$ and $\upsilon$ are the control vectors of the first and second players, respectively, $A(t), B(t)$ and $C(t)$ are piecewise-continuous matrix functions that are continuous from the right at points of discontinuity, and $t_0$ and $\vartheta$ are fixed instants of time. The admissible controls of the first and second players, $u[t_0[\cdot]\vartheta) = \{u[t], t_0 \leqslant t < \vartheta\}$ and $\vartheta[t_0[\cdot]\vartheta)$, respectively, are assumed to be any Borel-measurable vector functions that simultaneously satisfied the geometrical constraints

$$|u[t]|_1 \leqslant M, \ |\upsilon[t]|_2 \leqslant N, \quad t_0 \leqslant t < \vartheta \tag{1.2}$$

and the integral-momentum constraints

$$\int_{t_0}^{\vartheta} \alpha(\tau)|u[\tau]|_1 \, d\tau \leqslant \mu[t_0], \quad \int_{t_0}^{\vartheta} \beta(\tau)|\upsilon[\tau]|_2 \, d\tau \leqslant \nu[t_0] \tag{1.3}$$

Here $|\cdot|_1$, $|\cdot|_2$ are certain norms in $R^r$ and $R^s$ respectively, $M$ and $N$ are known constants, $\mu[t_0] > 0$, $\nu[t_0] > 0$ are specified numbers, and $\alpha(\tau)$ and $\beta(\tau)$ are positive scalar functions continuous in $[t_0, \vartheta]$.

We will consider the problem of the controls $u$ and $\upsilon$ that are respectively intended to minimize and maximize the terminal quality index

$$\gamma = |x[\vartheta]|_3 \tag{1.4}$$

where $|\cdot|_3$ is some norm in $R^n$.

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We will formalize this problem as a differential game [5, 8].

We supplement the phase vector $x$ of system (1.1) by introducing variables $\mu$ and $\nu$ whose evolution is described by the equations

$$d\mu/dt = -\alpha(t)|u|_1, \quad d\nu/dt = -\beta(t)|\upsilon|_2, \quad t_0 \leq t \leq \vartheta \tag{1.5}$$

Then the integral restrictions (1.3) are rewritten in the form of phase restrictions on $\mu$ and $\nu$

$$0 \leq \mu \leq \mu[t_0], \quad 0 \leq \nu \leq \nu[t_0] \tag{1.6}$$

By a strategy $u(\cdot)$ for the first player and $\upsilon(\cdot)$ for the second player we mean any vector functions

$$u(\cdot) = \{u(t, x, \mu, \nu, \varepsilon), \; |u(t, x, \mu, \nu, \varepsilon)|_1 \leq M, \; \{t, x, \mu, \nu\} \in G, \; \varepsilon > 0\} \tag{1.7}$$

$$\upsilon(\cdot) = \{\upsilon(t, x, \mu, \nu, \varepsilon), \; |\upsilon(t, x, \mu, \nu, \varepsilon)|_2 \leq N, \; \{t, x, \mu, \nu\} \in G, \; \varepsilon > 0\} \tag{1.8}$$

where

$$G = \{\{t, x, \mu, \nu\}: \; t_0 \leq t \leq \vartheta, \; |x|_e \leq R[t], \; 0 \leq \mu \leq \mu[t_0], \; 0 \leq \nu \leq \nu[t_0]\} \tag{1.9}$$

$$R[t] = (1 + R_0)\exp\{\lambda(t - t_0)\} - 1$$

is the domain of possible positions of the extended system (1.1), (1.5) and $\varepsilon$ is the precision parameter [5, 8]. In (1.9) $|x|_e$ is the Euclidean norm of $x$, and $R_0 > 0$ is a sufficiently large number determined from the initial state of system (1.1) at time $t = t_0$; the value of the parameter $\lambda$ is given by the formula

$$\lambda = \lambda_1 + \lambda_2 M\exp(\lambda_1(\vartheta - t_0)) + \lambda_3 N \tag{1.10}$$

$$\lambda_1 = \max_t\|A(t)\|, \quad \lambda_2 = \max_t\|B(t)\|, \quad \lambda_3 = \max_t\|C(t)\|, \quad t_0 \leq t \leq \vartheta$$

$$\|A(t)\| = \max_x|A(t)x|_e, \quad |x|_e \leq 1; \quad \|B(t)\| = \max_u|B(t)u|_e, \quad |u|_1 \leq 1$$

$$\|C(t)\| = \max_\upsilon|C(t)\upsilon|_e, \quad |\upsilon|_2 \leq 1$$

Note that the restriction on the actual value of the phase vector $x$ in the definition of the domain $G$ in (1.9) is not onerous because it is satisfied by any motion of the system (1.1) generated by arbitrary measurable realizations of controls that conform to the geometrical restrictions (1.2). However, restrictions (1.6) on the values of $\mu$ and $\nu$ (1.5) are not in general guaranteed by such realizations unless the additional conditions (1.3) are also assumed.

Suppose that the position $\{t_*, x_*, \mu_*, \nu_*\} \in G$ (1.9) has been reached. We denote by $\Delta_\delta$ ($\delta > 0$) a decomposition

$$\Delta_\delta = \{t_i\!: \; t_1 = t_*, \; t_{i+1} > t_i, \; t_{i+1} - t_i \leq \delta, \; i = 1,\ldots, k, \; t_{k+1} = \vartheta\} \tag{1.11}$$

of the interval $[t_*, \vartheta]$ in which we include all points of discontinuity of the matrix functions $A(t)$, $B(t)$ and $C(t)$. A chosen strategy $u(\cdot)$ for the first player, the value $\varepsilon > 0$ and the decomposition $\Delta_\delta$ (1.11) determine the control law $U$ of the first player

$$U = \{u(\cdot), \varepsilon, \Delta_\delta\} \tag{1.12}$$

Suppose that the system is acted upon by some admissible control

$$\upsilon[t_*[\cdot]\vartheta) = \{\upsilon[t], |\upsilon[t]|_2 \leq N, \quad t_* \leq t < \vartheta, \quad \int_{t_*}^{\vartheta} \beta(\tau)|\upsilon[\tau]|_2\, d\tau \leq \nu_*\} \tag{1.13}$$

of the second player. Then the motion $\{x_U[t], \mu_U[t], \nu[t]\}$, $t_* \leq t \leq \vartheta$, where the quantities $\mu_U[t]$, $\nu[t]$ give the remaining resources at time $t$ for the first and second players respectively, is uniquely shaped under the action of the law $U$ as the solution of the stepwise differential equations

$$dx_U[t]/dt = A(t)x_U[t] + B(t)u_{[t_i, t_{i+1})}[t] + C(t)\upsilon[t] \tag{1.14}$$

$$d\mu_U[t]/dt = -\alpha(t)|u_{[t_i,t_{i+1})}[t]|_1, \quad d\nu[t]/dt = -\beta(t)|\upsilon[t]|_2$$
$$t_i \leqslant t < t_{i+1}, \quad i = 1,\dots,k; \quad x_U[t_1] = x_*, \quad \mu_U[t_1] = \mu_*, \quad \nu[t_1] = \nu_*$$

The initial state $\{x_U[t_i], \mu_U[t_i], \nu[t_i]\}$ for $t_i \leqslant t \leqslant t_{i+1}, i > 1$ is identical with the final state $\{x_U[t_i], \mu_U[t_i], \nu[t_i]\}$ for the preceding interval $t_{i-1} \leqslant t \leqslant t_i$. In (1.14) the realization of the control of the first player $u_{[t_i,t_{i+1})}[t], t_i \leqslant t < t_{i+1}, i = 1,\dots,k$ at each step is designated by the law $U$ (1.12) in the following fashion

$$u_{[t_i,t_{i+1})}[t] = 0, \quad t_i \leqslant t < t_{i+1}, \quad \text{if} \quad \mu_U[t_i] = 0 \tag{1.15}$$

$$u_{[t_i,t_{i+1})}[t] = \begin{cases} u_i, & t_i \leqslant t < t', \\ 0, & t' \leqslant t < t_{i+1}, \end{cases} \quad \text{if} \quad 0 < \mu_U[t_i] < J_i(t_{i+1})$$

where $t'$ is defined by the condition $J_i(t') = \mu_U[t_i]$; we finally have

$$u_{[t_i,t_{i+1})}[t] = u_i, \quad t_i \leqslant t < t_{i+1}, \quad \text{if} \quad \mu_U[t_i] \geqslant J_i(t_{i+1})$$

Here

$$u_i = u(t_i, x_U[t_i], \ \mu_U[t_i], \ \nu[t_i], \varepsilon), \quad J_i(t) = \int_{t_i}^{t} \alpha(\tau)|u_i|_1 \, d\tau$$

Similarly, given that the position $\{t_*, x_*, \mu_*, \nu_*\} \in G$ (1.9) has been reached, a chosen strategy $\upsilon(\cdot)$ (1.8) for the second player, the value of $\varepsilon > 0$ and the decomposition $\Delta_\delta$ determine the control law $V = \{\upsilon(\cdot), \varepsilon, \Delta_\delta\}$ of the second player. This law, paired with some admissible realization of the control of the first player

$$u[t_*[\cdot]\vartheta) = \{u[t], |u[t]|_1 \leqslant M, \quad t_* \leqslant t < \vartheta, \quad \int_{t_*}^{\vartheta} \alpha(\tau)|u[\tau]|_1 \, d\tau \leqslant \mu_*\} \tag{1.16}$$

uniquely defines the motion $\{x_V[t], \mu[t], \nu_V[t]\}, t_* \leqslant t \leqslant \vartheta$.

The realizations of the controls $u[t_*[\cdot]\vartheta) = \{u_{[t_i,t_{i+1})}[t], t_i \leqslant t < t_{i+1}, i = 1,\dots,k\}$ and $\upsilon[t_*[\cdot]\vartheta] = \{\upsilon_{[t_i,t_{i+1})}[t], t_i \leqslant t < t_{i+1}, i = 1,\dots,k\}$ shaped by the laws $U$ and $V$ and constructed by (1.15) (for $u[\cdot]$, and similarly for $\upsilon[\cdot]$) are admissible in the sense of (1.16) and (1.13). Consequently, by virtue of (1.9) and (1.10) the positions $\{t, x_U[t], \mu_U[t], \nu[t]\}$ and $\{t, x_V[t], \mu[t], \nu_V[t]\}, t_* \leqslant t \leqslant \vartheta$ considered do not leave the confines of the domain $G$.

The quantities

$$\rho_u(u(\cdot); \ t_*, x_*, \mu_*, \nu_*) = \overline{\lim_{\varepsilon\to 0}} \lim_{\delta\to 0} \sup_{\Delta_\delta} \sup_{\upsilon[\cdot]} \gamma \tag{1.17}$$

$$\rho_\upsilon(\upsilon(\cdot); \ t_*, x_*, \mu_*, \nu_*) = \underline{\lim_{\varepsilon\to 0}} \lim_{\delta\to 0} \inf_{\Delta_\delta} \inf_{u[\cdot]} \gamma \tag{1.18}$$

are called the guaranteed results of the strategies $u(\cdot)$, $\upsilon(\cdot)$ and the original position $\{t_*, x_*, \mu_*, \nu_*\} \in G$. Here $\Delta_\delta$ is the decomposition (1.11); in (1.17) and (1.18) the upper and lower bounds are taken over all measurable realizations $u[t_*[\cdot]\vartheta)$ (1.16), $\upsilon[t_*[\cdot]\vartheta)$ (1.13).

We say that the first player strategy $u^0(\cdot)$ and the second player strategy $\upsilon^0(\cdot)$ are optimal if for any position $\{t_*, x_*, \mu_*, \nu_*\} \in G$ we have

$$\rho_u(u^0(\cdot); \ t_*, x_*, \mu_*, \nu_*) = \min_{u(\cdot)} \rho_u(u(\cdot); \ t_*, x_*, \mu_*, \nu_*)$$

$$\rho_\upsilon(\upsilon^0(\cdot); \ t_*, x_*, \mu_*, \nu_*) = \max_{\upsilon(\cdot)} \rho_\upsilon(\upsilon(\cdot); \ t_*, x_*, \mu_*, \nu_*)$$

The existence of optimal strategies $u^0(\cdot)$, $\upsilon^0(\cdot)$ and the validity of the equality

$$\rho_u(u^0(\cdot); t_*, x_*, \mu_*, \nu_*) = \rho_\upsilon(\upsilon^0(\cdot); t_*, x_*, \mu_*, \nu_*) = \rho^0(t_*, x_*, \mu_*, \nu_*)$$

for any position $\{t_*, x_*, \mu_*, \nu_*\} \in G$ were established in [8]. This means that $\rho^0(t_*, x_*, \mu_*, \nu_*)$ is the value of the differential game under consideration, and the pair of optimal solutions $\{u^0(\cdot), \upsilon^0(\cdot)\}$ forms a saddle point. We stress that optimal strategies are efficiently constructed from a known value function by the method of extremal displacement to a comoving position [8].

## 2. CALCULATION OF THE VALUE OF THE GAME

We consider the auxiliary model

$$dw/d\tau = A(\tau)w + B(\tau)u + C(\tau)\upsilon, \quad dw_{n+1}/d\tau = -\alpha(\tau)|u|_1$$
$$dw_{n+2}/d\tau = -\beta(\tau)|\upsilon|_2, \quad w \in R^n, \quad u \in R^r, \quad \upsilon \in R^s, \quad t_0 \leqslant \tau \leqslant \vartheta \tag{2.1}$$

We define the domain $G'$ of possible positions for the model

$$G' = \{\{\tau, w, w_{n+1}, w_{n+2}\}: \ t_0 \leqslant \tau \leqslant \vartheta, \quad |w|_c \leqslant R'[\tau], \quad 0 \leqslant w_{n+1} \leqslant \mu[t_0] + \xi,$$
$$0 \leqslant w_{n+2} \leqslant \nu[t_0] + \xi\}, \quad R'[\tau] = (1 + R_0 + \xi)\exp\{\lambda(\tau - t_0)\} - 1 \tag{2.2}$$

In (2.2) $\xi > 0$ is a fairly small fixed number, and $R_0$ and $\lambda$ are given by (1.9) and (1.10). We note that $G$ (1.9) $\subset G'$ (2.2) ($\{\tau, w, w_{n+1}, w_{n+2}\} = \{t, x, \mu, \nu\}$), and also that the motions $\{w[\tau_*[\cdot]\vartheta], w_{n+1}[\tau_*[\cdot]\vartheta], w_{n+2}[\tau_*[\cdot]\vartheta]\}$ of model (2.1) generated from any position $\{\tau_*, w_*, w_{*n+1}, w_{*n+2}\} \in G'$ by realizations $u[\tau_*[\cdot]\vartheta)$ (1.16), $\upsilon[\tau_*[\cdot]\vartheta)$ (1.13), where $t_* = \tau_*$, $\mu_* = w_{*n+1}$, $\nu_* = w_{*n+2}$, do not leave the domain $G'$.

Suppose that a position $\{\tau_*, w_*, \mu_*, \nu_*\} \in G'$ (2.2) for model (2.1) has been chosen, together with a decomposition $\Delta_k = \Delta_k\{\tau_j\} = \{\tau_j: \tau_1 = \tau_*, \tau_{j+1} > \tau_j, j = 1, \ldots, k, \tau_{k+1} = \vartheta\}$ of the interval $[\tau_*, \vartheta]$ which includes all points of discontinuity of the matrix functions $A(t)$, $B(t)$ and $C(t)$. We introduce the functions

$$\Delta\psi_j(m, \mu, \nu) = \min_{u[\cdot]\in D_j^u(\mu)} \max_{\upsilon[\cdot]\in D_j^\upsilon(\nu)} \left[\int_{\tau_j}^{\tau_{j+1}} \langle m, X[\vartheta, \tau](B(\tau)u[\tau] + C(\tau)\upsilon[\tau])\rangle d\tau\right] \tag{2.3}$$
$$D_j^u(\mu) = \left\{u[\tau]: |u[\tau]|_1 \leqslant M, \quad \tau_j \leqslant \tau < \tau_{j+1}; \quad \int_{\tau_j}^{\tau_{j+1}} \alpha(\tau)|u[\tau]|_1\, d\tau \leqslant \mu\right\}$$
$$D_j^\upsilon(\nu) = \left\{\upsilon[\tau]: |\upsilon[\tau]|_2 \leqslant N, \quad \tau_j \leqslant \tau < \tau_{j+1}; \quad \int_{\tau_j}^{\tau_{j+1}} \beta(\tau)|\upsilon[\tau]|_2\, d\tau \leqslant \nu\right\}$$
$$j = 1, \ldots, k, \quad \mu \geqslant 0, \quad \nu \geqslant 0; \quad m \in S = \{m \in R^n, |m|^* \leqslant 1\}$$

where $|\cdot|^*$ is the norm conjugate to the norm $|\cdot|_3$ which appears in (1.4); $X[t, \tau]$ is the fundamental matrix for the equation $dx/dt = A(t)x$, and we denote by $\langle\cdot,\cdot\rangle$ the scalar vector product; $D_j^u(\mu)$ and $D_j^\upsilon(\nu)$ are weakly-compact sets of measurable vector functions. The function $\Delta\psi_j(m, \mu, \nu)$ is continuous over the set of its arguments, convex in $\mu$, non-increasing in $\mu$ and non-decreasing in $\nu$. The values of $\Delta\psi_j(\cdot)$ (2.3) do not change if in the definition of the sets $D_j^u(\mu)$ and $D_j^\upsilon(\nu)$ the inequality sign in the last integral constraints is changed to an equality sign (for those values of $\mu$ and $\nu$ for which this equality is possible).

We then introduce the function $\varphi_j^*(\cdot)$ by the recursive procedure

$$\varphi_{k+1}^*(m, \mu, \nu) = 0, \quad \varphi_j^*(m, \mu, \nu) = \{\psi_j^*(\cdot, \mu, \nu)\}_* \tag{2.4}$$
$$\psi_j^*(m, \mu, \nu) = \max_{\nu' \in N_j(\nu)} \min_{\mu' \in M_j(\mu)} [\Delta\psi_j(m, \mu - \mu', \nu - \nu') + \varphi_{j+1}^*(m, \mu', \nu')]$$
$$m \in S, \quad 0 \leqslant \mu \leqslant \mu_*, \quad 0 \leqslant \nu \leqslant \nu_*, \quad j = k, \ldots, 1$$

where

$$M_j(\mu)=\left\{\mu':\ \max\left[0,\ \mu-M\int\limits_{\tau_j}^{\tau_{j+1}}\alpha(\tau)d\tau\right]\leqslant\mu'\leqslant\mu\right\} \tag{2.5}$$

$$N_j(\nu)=\left\{\nu':\ \max\left[0,\ \nu-N\int\limits_{\tau_j}^{\tau_{j+1}}\beta(\tau)d\tau\right]\leqslant\nu'\leqslant\nu\right\}$$

and the quantity

$$e^*(\tau_*,w_*,\mu_*,\nu_*;\Delta_k)=\max_{m\in S}[\langle m,X[\vartheta,\tau_*]w_*\rangle+\varphi_1^*(m,\mu_*,\nu_*)] \tag{2.6}$$

In (2.4) the symbol $\varphi(m) = \{\psi(\cdot)\}_*$ denotes the upper convex envelope of the function $\psi(\cdot)$—the minimal concave function which majorizes the function $\psi(m)$, $m \in S$. Here and below the upper convex envelope is taken only for $m \in S$ for fixed values of the other variables. In (2.4) and (2.6) the corresponding minima and maxima are indeed reached since the following lemma holds.

*Lemma* 1. If $S = \{m \in R^n, \mid m \mid^* \leqslant 1\}$ is a polyhedron or a strictly convex set, then the functions $\psi_j^*(m, \mu, \nu)$ and $\varphi_j^*(m, \mu, \nu), j = k, \ldots, 1$ are continuous on the set $S \times [0, \mu_*] \times [0, \nu_*]$, convex in $\mu$, non-increasing in $\mu$ and non-decreasing in $\nu$, with $\varphi_j^*(m, \mu, \nu)$ concave in $m$.

The validity of Lemma 1 follows from the well-known facts of convex analysis [9].

*Remark.* If some function $\psi(m)$, $m \in S$ is continuous, then $\{\psi(m)\}_*$ will always be continuous on the entire set $S$ in the case when $S$ satisfies the conditions of Lemma 1, from which the validity of the above follows. If not, one can choose $S$ to be a polyhedron approximating the set $\{m \in R^n, \mid m \mid^* \leqslant 1\}$. Then the further arguments are unchanged, and the result obtained will be true with an accuracy corresponding to the accuracy of the given approximation.

We recall that when the value of the game was calculated in [8] the functions $\psi_j(m, \mu, \nu)$ and $\varphi_j(m, \mu, \nu)$ were used instead of the functions $\psi_j^*(m, \mu, \nu)$ and $\varphi_j^*(m, \mu, \nu)$, and that the former were determined in terms of $\Delta\psi_j(m, \mu, \nu)$ (2.3) using formulae (2.4), (2.5), with the sole difference that in (2.4) the operation $\min_{\mu'}\max_{\nu'}$ appeared instead of $\max_{\nu'}\min_{\mu'}$. Correspondingly, instead of $e^*(\tau_*, w_*, \mu_*, \nu_*; \Delta_k)$ in accordance with (2.6), where on the right-hand side one has to replace $\varphi_1^*(\cdot)$ with $\varphi_1(\cdot)$, the quantity $e(\tau_*, w_*, \mu_*, \nu_*; \Delta_k)$ was defined, for which the following result was established [8]

*Theorem* 1. Whatever the original position $\{t_*, x_*, \mu_*, \nu_*\} \in G$ (1.9) and sequence of subdivisions $\Delta_k = \Delta_k\{\tau_j\}$ $(k = 1, 2, \ldots)$ of the interval $[t_*, \vartheta]$, $\tau_1 = t_*$, $\tau_{k+1} = \vartheta$ with step $\delta_k = \max_j(\tau_{j+1} - \tau_j)$, $j = 1, \ldots, k$ satisfying the condition $\lim \delta_k = 0$, $k \to \infty$, we have the equality

$$\lim_{k\to\infty} e(t_*,x_*,\mu_*,\nu_*;\Delta_k)=\rho^0(t_*,x_*,\mu_*,\nu_*)$$

In view of the definition if $e^*(\cdot)$ (2.6) and $e(\cdot)$ in [8], for any positions $\{\tau_*, w_*, \mu_*, \nu_*\}$ and subdivision $\Delta_k$ of the interval $[\tau_*, \vartheta]$, we have the inequality

$$e^*(\tau_*, w_*, \mu_*, \nu_*; \Delta_k) \leqslant e(\tau_*, w_*, \mu_*, \nu_*; \Delta_k) \tag{2.7}$$

Furthermore, a situation may arise in which inequality (2.7) is strict.

This is shown by the following

*Example.* Consider the system described by the equations

$$dx/dt = b(t)u + c(t)\upsilon,\quad 0 \leqslant t \leqslant 2,\quad x \in R^1,\quad u \in R^1,\quad \upsilon \in R^1 \tag{2.8}$$

where

$$b(t)=\begin{cases}2, & 0\leqslant t<1\\ 1, & 1\leqslant t\leqslant 2\end{cases};\quad c(t)=\begin{cases}6, & 0\leqslant t<1\\ 2, & 1\leqslant t\leqslant 2\end{cases} \tag{2.9}$$

Suppose that the controls $u$ and $\upsilon$ are constrained by restrictions of mixed type

$$|u| \leq 1, \quad |\upsilon| \leq 1; \quad \int_0^2 |u[\tau]| d\tau \leq 1, \quad \int_0^2 |\upsilon[\tau]| d\tau \leq 0.25 \qquad (2.10)$$

and the quality index has the form

$$\gamma = |x[2]| \qquad (2.11)$$

Suppose that the position $\{\tau_*, w_*, \mu_*, \nu_*\} = \{0, 0.5, 1, 0.25\}$ has occurred. We assign the subdivision $\Delta_2 = \{\tau_1 = 0, \tau_2 = 1, \tau_3 = 2\}$ to the time interval $[0, 2]$.

Then, performing the calculations, we obtain

$$\varphi_1^*(m, 1, 0,25) = -|m|/2, \quad \varphi_1(m, 1, 0.25) = \begin{cases} -|m|^2/(1-|m|), & |m| \leq 0,2 \\ (1-9|m|)/16, & 0,2 < |m| \leq 1 \end{cases} \qquad (2.12)$$

and correspondingly

$$e^*(0, 0.5, 1, 0.25; \Delta_2) = 0, \quad e(0, 0.5, 1, 0.25; \Delta_2) = 2.5 - \sqrt{6} > 0 \qquad (2.13)$$

We show below that despite inequality (2.7) (possibly strict, as, for example, in (2.8)–(2.13)) for any positions $\{t_*, x_*, \mu_*, \nu_*\} \in G$ (1.9) of the extended system (1.1), (1.5) and sequence of subdivisions $\Delta_k = \Delta_k\{\tau_j\}$ $(k = 1, 2, \ldots)$ of the interval $[t_*, \vartheta]$, $\tau_1 = t_*$, $\tau_{k+1} = \vartheta$ with step $\delta_k = \max_j(\tau_{j+1} - \tau_j)$, $j = 1, \ldots, k$ satisfying the condition $\lim \delta_k = 0$, $k \to \infty$, we have the equality

$$\lim_{k\to\infty} e^*(t_*, x_*, \mu_*, \nu_*; \Delta_k) = \lim_{k\to\infty} e(t_*, x_*, \mu_*, \nu_*; \Delta_k) = \rho^0(t_*, x_*, \mu_*, \nu_*) \qquad (2.14)$$

We first establish the important properties of $\mu$-stability ([5], p. 208) and $\upsilon$-stability [5, p. 216] of the quantity $e^*(\cdot)$ (2.6).

*Theorem* 2 ($\mu$-stability of $e^*(\cdot)$). For any $\varepsilon > 0$ a $\delta(\varepsilon) > 0$ exists such that whatever the position of the model $\{\tau_*, w_* = w[\tau_*], \mu_*[\tau_*], \nu_* = \nu[\tau_*]\} \in G'$ (2.2) $(w_{n+1} = \mu, w_{n+2} = \nu)$ and subdivision $\Delta_k = \Delta_k\{\tau_j\}$ of the interval $[\tau_*, \vartheta]$, for any admissible realization $\upsilon_*[\tau_*[\cdot]\tau^*) \in D_1^{\upsilon}(\nu_*)$ (2.3), where $\tau^* = \tau_2$ is the second point of the subdivision $\Delta_k\{\tau_j\}$, one can find an admissible realization $u[\tau_*[\cdot]\tau^*) \in D_1^u(\mu_*)$ (2.3) such that the motion of the model generated by these realizations from the position $\{\tau_*, w_*, \mu_*, \nu_*\}$ arrives at the position $\{\tau^*, w^* = w[\tau^*], \mu^* = \mu[\tau^*], \nu^* = \nu^*[\tau^*]\} \in G'$, and the inequality

$$e^*(\tau^*, w^*, \mu^*, \nu^*; \Delta_{k^*}^*) - e^*(\tau_*, w_*, \mu_*, \nu_*; \Delta_k) \leq \varepsilon(\tau^* - \tau_*) \qquad (2.15)$$

is satisfied if only

$$\tau^* - \tau_* \leq \delta(\varepsilon) \qquad (2.16)$$

where $\Delta_{k^*}^* = \Delta_{k^*}^*\{\tau_j^*\}$ is a subdivision of the interval $[\tau^*, \vartheta]$, $\tau_1^* = \tau^*$, $\tau_{k^*+1}^* = \vartheta$ satisfying the condition $\tau_j^* = \tau_i$, $i = j + 1$, $j = 1, \ldots, k^*$, $k^* = k - 1$, and $\tau_j \in \Delta_k$.

*Proof.* Suppose that position $\{\tau_*, w_*, \mu_*, \nu_*\} \in G'$ has occurred and that a subdivision $\Delta_k$ of the interval $[\tau_*, \vartheta]$ has been chosen satisfying condition (2.16) with a $\delta(\varepsilon) > 0$ sufficiently small for the following arguments to be valid. We fix a realization $\upsilon_*[\tau_*[\cdot]\tau^*) \in D_1^{\upsilon}(\nu_*)$, and consequently $\nu^* = \nu[\tau^*]$ as well. We consider the set $MW = MW(\tau^*, \tau_*, w_*, \mu_*) \subset R^{n+1}$—the domain of accessibility in the space of variables $\{w, \mu\} \subset R^{n+1}$ up to time $\tau^*$ for motions of the model generated from the position $\{\tau_*, w_*, \mu_*, \nu_*\}$ by any control $u[\cdot] \in D_1^u(\mu_*)$ paired up with $\upsilon_*[\tau_*[\cdot]\tau^*)$. This set is non-empty, convex and compact in $R^{n+1}$. Furthermore, if $y = \{w, \mu\} \in MW$, then $\mu \in M_1(\mu_*)$ (2.5) and for any $\mu' \leq \mu$, $\mu' \in M_1(\mu_*)$ we have $y' = \{w, \mu'\} \in MW$. Along with $MW$ we introduce the set

$$MW' = MW'(\tau^*, \tau_*, w_*, \mu_*) \subset R^{n+1}$$

$$MW' = \left\{\{w', \mu\}: \ \mu = \mu_* - \int_{\tau_*}^{\tau^*} \alpha_1 |u[\tau]|_1 \, d\tau, \quad \mu \in M_1(\mu_*)\right.$$

$$w' = X[\tau^*, \tau_*]w_* + \int_{\tau_*}^{\tau^*} X[\tau^*, \tau]C(\tau)\upsilon_*[\tau]d\tau + \int_{\tau_*}^{\tau^*} X[\tau^*, \tau_*]B(\tau_*)u[\tau]d\tau$$

$$|u[\tau]|_1 \leqslant M, \quad \alpha_1 = \max_\tau \alpha(\tau), \quad \tau_* \leqslant \tau \leqslant \tau^* \Big\} \tag{2.17}$$

The set $MW'$ (2.17) is convex and compact in $R^{n+1}$. The intersection of $MW'$ and the hyperplane $\mu = \mu^0$ (which we denote by $MW' \mid \mu = \mu^0$) is non-empty and compact in $R^n$ for all $\mu^0 \in M_1(\mu_*)$. We also note that if $y = \{w', \mu\} \in MW'$, then by (1.10), (2.2) and (2.17) the position $\{\tau^*, w', \mu, \nu^*\} \in G'$ (2.2). Because of these properties of the sets $MW$ and $MW'$ and the continuity properties of $A(\tau)$, $B(\tau)$, $\alpha(\tau)$ when $\tau_* \leqslant \tau \leqslant \tau^*$ (recalling that all the points of discontinuity of the matrix functions $A(t)$ and $B(t)$ are included in the decomposition $\Delta_k$), we have the following lemma.

*Lemma* 2. For any $\varepsilon > 0$ a $\delta(\varepsilon) > 0$ exists such that for any vector $y^0 = \{w'^0, \mu^0\} \in MW'$ one can find a vector $z^0 = \{w^0, \mu^0\} \in MW$ such that the relation

$$|y^0 - z^0|_e = |w'^0 - w^0|_e \leqslant \varepsilon(\tau^* - \tau_*)$$

holds so long as condition (2.16) is satisfied.

To prove the theorem it is sufficient to show that a vector $y^0 = \{w'^0, \mu^0\} \in MW'$ exists satisfying the condition

$$\Delta e^* = e^*(\tau^*, w'^0, \mu^0, \nu^*; \Delta^*_{k^*}) - e^*(\tau_*, w_*, \mu_*, \nu_*; \Delta_k) \leqslant \varepsilon(\tau^* - \tau_*) \tag{2.18}$$

where $\tau^* - \tau_* \leqslant \delta(\varepsilon)$ and $\delta(\varepsilon) > 0$ is a sufficiently small number.

Indeed, according to Lemma 2, it follows from (2.18) and the Lipschitz property of $e^*(\tau, w, \mu, \nu; \Delta)$ (2.6) with respect to the variable $w$ that a $z^0 = \{w^0, \mu^0\} \in MW$ exists such that inequality (2.15) (with $w^* = w^0$, $\mu^* = \mu^0$) holds. Bearing in mind the definition of the set $MW$ and the domain $G'$ (2.2), that is the content of the theorem.

We then suppose that a pair $(m^0, y^0 = \{w'^0, \mu^0\}) \in D = [S \times MW']$ has been found which simultaneously satisfies three conditions

$$\langle m^0, X[\vartheta, \tau^*]w'^0\rangle + \varphi_2^*(m^0, \mu^0, \nu^*) = \max_{m \in S}[\text{Idem}(m^0 \to m)] \tag{2.19}$$

$$\Delta\psi_1(m^0, \mu_* - \mu^0, \nu_* - \nu^*) + \varphi_2^*(m^0, \mu^0, \nu^*) = \min_{\mu' \in M_1(\mu_*)}[\text{Idem}(\mu^0 \to \mu')] \tag{2.20}$$

$$\langle m^0, X[\vartheta, \tau^*]w'^0\rangle = \min_{w' \in MW' \mid \mu = \mu^0}[\langle m^0, X[\vartheta, \tau^*]w'\rangle] \tag{2.21}$$

Here Idem on the right-hand side of an equation denotes the expression on the left-hand side with the substitution shown in the parentheses.

Suppose that $u^0[\tau_*[\cdot]\tau^*) = \{u^0[\tau], \tau_* \leqslant \tau < \tau^*\}$ is a realization of the control which, by (2.17), corresponds to the vector $y^0 = \{w'^0, \mu^0\} \in MW'$. From (2.17) and (2.21) we have

$$\int_{\tau_*}^{\tau^*} \langle m^0, X[\vartheta, \tau_*]B(\tau_*)u^0[\tau]\rangle d\tau = \min_{u[\cdot]}[\text{Idem}(u^0[\tau] \to u[\tau])] \tag{2.22}$$

$$|u[\tau]|_1 \leqslant M, \quad \tau_* \leqslant \tau < \tau^*, \quad \int_{\tau_*}^{\tau^*} \alpha_1 |u[\tau]|_1 \, d\tau = \mu_* - \mu^0$$

Then, by virtue of (2.6), (2.17) and (2.19) we obtain

$$e^*(\tau^*, w'^0, \mu^0, \nu^*; \Delta^*_{k^*}) = \langle m^0, X[\vartheta, \tau^*]w'^0\rangle + \varphi_2^*(m^0, \mu^0, \nu^*) =$$

$$= \langle m^0, X[\vartheta, \tau_*]w_*\rangle + \int_{\tau_*}^{\tau^*} \langle m^0, X[\vartheta, \tau]C(\tau)\upsilon_*[\tau]\rangle d\tau +$$

$$+ \int_{\tau_*}^{\tau^*} \langle m^0, X[\vartheta, \tau_*]B(\tau_*)u^0[\tau]\rangle d\tau + \varphi_2^*(m^0, \mu^0, \nu^*) \tag{2.23}$$

On the other hand, from (2.4)–(2.6), (2.19) and (2.20) we have

$$e^*(\tau_*, w_*, \mu_*, \nu_*; \Delta_k) \geqslant \langle m^0, X[\vartheta, \tau_*]w_*\rangle + \varphi_1^*(m^0, \mu_*, \nu_*) \geqslant$$
$$\geqslant \langle m^0, X[\vartheta, \tau_*]w_*\rangle + \Delta\psi_1(m^0, \mu_* - \mu^0, \nu_* - \nu^*) + \varphi_2^*(m^0, \mu^0, \nu^*) \quad (2.24)$$

The validity of Eq. (2.18) follows from (2.23) and (2.24), using (2.3) and (2.22) and the continuity properties of $A(\tau)$, $B(\tau)$, $\alpha(\tau)$ when $\tau_* \leqslant \tau \leqslant \tau_*$.

We now verify that the pair $(m^0, y^0 = \{w'^0, \mu^0\})$ (2.19)–(2.21) does indeed exist. We construct a mapping $D \to D$. We put each pair $(m, y) \in D$ in correspondence with the set $D_1[m, y]$ of all possible pairs $(m^{(1)}, y^{(2)})$, where $m^{(1)} \in M^0(\tau^*, y = \{w', \mu\}) \subset S$, and $y^{(1)} \in MW'^0(m) \subset MW'$ (2.17). Here $M^0(\tau^*, y = \{w', \mu\})$ is the set of maximizing vectors $m^0$ satisfying (2.19) with $w'^0 = w'$, $\mu^0 = \mu$, and $MW'^0(m)$ is the set of vectors $y^0 = \{w'^0, \mu^0\}$ satisfying conditions (2.20), (2.21) with $m^0 = m$. The minimum on the right-hand side of (2.21) ($m^0 \to m$) is a function continuous in its set of variables $\{m, \mu^0\}$, $m \in S$, $\mu^0 \in M_1(\mu_*)$ (2.5) and linear in $\mu^0$, which follows from direct calculations of this minimum using (2.17). Using this and Lemma 1, one can verify that the sets $M^0(\tau^*, y = \{w', \mu\})$ are non-empty, convex, compact in $R^n$ and varying semicontinuously from above by inclusion as $y$ changes, and the sets $MW'^0(m)$ are non-empty, convex, compact in $R^{n+1}$ and vary semicontinuously from above by inclusion as $m$ changes. Hence, by Kakutani's theorem [10, p. 638], the mapping in question has a fixed point $(m^0, y^0 = \{w'^0, \mu^0\}) \in D$ for which conditions (2.19)–(2.22) are simultaneously satisfied. Theorem 2 is proved.

*Theorem* 3 ($\upsilon$-stability of $e^*(\cdot)$). Suppose that the position of model (2.1) $\{\tau_*, w_* = w[\tau_*], \mu_* = \mu[\tau_*], \nu_* = \nu[\tau_*]\} \in G'$ (2.2) $w_{n+1} = \mu$, $w_{n+2} = \nu$ and the subdivision $\Delta_k = \Delta_k\{\tau_j\}$ of the interval $[\tau_*, \vartheta]$ have been chosen. Then one can find an admissible realization $\upsilon_0[\tau_*[\cdot]\tau^*) \in D_1^{\upsilon}(\nu_*)$ (2.3) where $\tau^* = \tau_2 \in \Delta_k$ such that for any realization $u[\tau_*[\cdot]\tau^*) \in D_1^u(\mu_*)$ (2.3) the inequality

$$e^*(\tau^*, w[\tau^*], \mu[\tau^*], \nu[\tau^*]; \Delta_{k^*}^*) - e^*(\tau_*, w_*, \mu_*, \nu_*; \Delta_k) \geqslant 0 \quad (2.25)$$

holds for the corresponding motion of the model generated from the position $\{\tau_*, w_*, \mu_*, \nu_*\}$ by these controls.

*Proof.* By virtue of (2.4)–(2.6), Lemma 1 and known properties of envelopes convex from above which are corollaries of Carathéodory's theorem [9, p. 171], one finds a vector $m_0 \in S$ and a number $\nu_0 \in N_1(\nu_*)$ (2.5) such that for any realization $u[\tau_*[\cdot]\tau^*) \in D_1^u(\mu_*)$ we have the relations

$$e^*(\tau_*, w_*, \mu_*, \nu_*; \Delta_k) = \max_{m \in S}[\langle m, X[\vartheta, \tau_*]w_*\rangle + \{\psi_1^*(m, \mu_*, \nu_*)\}_*] =$$
$$= \langle m_0, X[\vartheta, \tau_*]w_*\rangle + \psi_1^*(m_0, \mu_*, \nu_*) \leqslant$$
$$\leqslant \langle m_0, X[\vartheta, \tau_*]w_*\rangle + \Delta\psi_1(m_0, \mu_* - \mu[\tau^*], \nu_* - \nu_0) + \varphi_2^*(m_0, \mu[\tau^*], \nu_0) \quad (2.26)$$

Suppose that a realization $\upsilon_0[\tau_{*[\cdot]}\tau^*)$ has been chosen

$$\int_{\tau_*}^{\tau^*} \langle m_0, X[\vartheta, \tau]C(\tau)\upsilon_0[\tau]\rangle d\tau = \max_{\upsilon[\cdot]}[\mathrm{Idem}(\upsilon_0[\tau] \to \upsilon[\tau])] \quad (2.27)$$

$$|\upsilon[\tau]|_2 \leqslant N, \quad \tau_* \leqslant \tau < \tau^*, \quad \int_{\tau_*}^{\tau^*} \beta(\tau)|\upsilon[\tau]|_2\, d\tau = \nu_* - \nu_0$$

Such a measurable realization $\upsilon_0[\cdot]$ exists from the construction $\upsilon_0[\cdot] \in D_1^u(\nu_*)$, and $\upsilon_0[\cdot]$ is universal, i.e. does not depend on the choice of the realization $u[\tau_*[\cdot]\tau^*) \in D_1^u(\mu_*)$. Then, under the action of the controls $u[\tau]$ and $\upsilon_0[\tau]$, $\tau_* \leqslant \tau < \tau^*$, the model arrives at the position $\{\tau^*, w[\tau^*], \mu[\tau^*], \nu[\tau^*]\} \in G'$ (2.2) for which, by (2.1), (2.6) and (2.27), we have

$$\nu[\tau^*] = \nu_0, \quad e^*(\tau^*, w[\tau^*], \mu[\tau^*], \nu[\tau^*]; \Delta_{k^*}^*) \geqslant$$
$$\geqslant \langle m_0, X[\vartheta, \tau^*]w[\tau^*]\rangle + \varphi_2^*(m_0, \mu[\tau^*], \nu[\tau^*]) = \langle m_0, X[\vartheta, \tau_*]w_* +$$
$$+ \int_{\tau_*}^{\tau^*} \langle m_0, X[\vartheta, \tau](B(\tau)u[\tau] + C(\tau)\upsilon_0[\tau])\rangle d\tau + \varphi_2^*(m_0, \mu[\tau^*], \nu_0) \quad (2.28)$$

Inequality (2.25) follows from (2.26)–(2.28) using (2.3). Theorem 3 is proved.

The existence of a value for the game under investigation means that from Theorems 2 and 3 we derive the following theorem as a corollary.

*Theorem* 4. Whatever the original position $\{t_*, x_*, \mu_*, \nu_*\} \in G$ (1.9) and sequence of decompositions $\Delta_k = \Delta_k\{\tau_j\}$ $(k = 1, 2, \ldots)$ of the interval $[t_*, \vartheta]$, $\tau_1 = t_*$, $\tau_{k+1} = \vartheta$ with step $\delta_k = \max_j(\tau_{j+1} - \tau_j)$, $j = 1, \ldots, k$ satisfying the condition $\lim \delta_k = 0$, $k \to \infty$, we have

$$\lim_{k\to\infty} e^*(t_*, x_*, \mu_*, \nu_*; \Delta_k) = \rho^0(t_*, x_*, \mu_*, \nu_*)$$

Equation (2.14) follows from Theorems 1 and 4 using the uniqueness of the value of the game.

In conclusion we note that by virtue of (2.14) and appropriate properties of $e^*(\cdot)$ and $e(\cdot)$, but $e^*(\cdot)$ and $e(\cdot)$ can be used to calculate approximately the value of the game and to construct strategies guaranteeing a result close to the value of the game to a previously specified level of accuracy. A description of the method of forming such strategies based on $e(\cdot)$ is given in [8]. Moreover, in this context the operations of minimizing and maximizing with respect to the resource parameters $\mu'$ and $\nu'$ in (2.4) can be interchanged, which is often useful in practical calculations.

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